Preprint typeset in JHEP style - HYPER VERSION

SNUTP 03-027
hep-th/0312103

Seiberg-Witten-type Maps for Currents and Energy-Momentum Tensors in Noncommutative Gauge Theories

Rabin Banerjee[a *†], **Choonkyu Lee**[b ‡] **and Hyun Seok Yang**[b §]

[a]*BK21 Physics Research Division and Institute of Basic Science, Sungkyunkwan University, Suwon 440-746, Republic of Korea*
[b]*School of Physics and Center for Theoretical Physics, Seoul National University, Seoul 151-747, Republic of Korea*

Abstract: We derive maps relating the currents and energy-momentum tensors in noncommutative (NC) gauge theories with their commutative equivalents. Some uses of these maps are discussed. Especially, in NC electrodynamics, we obtain a generalization of the Lorentz force law. Also, the same map for anomalous currents relates the Adler-Bell-Jackiw type NC covariant anomaly with the standard commutative-theory anomaly. For the particular case of two dimensions, we discuss the implications of these maps for the Sugawara-type energy-momentum tensor.

Keywords: Duality in Gauge Field Theories, Gauge Symmetry, Space-Time Symmetries.

[*]rabin@newton.skku.ac.kr
[†]On leave from S. N. Bose National Center for Basic Sciences, Calcutta, India; rabin@bose.res.in
[‡]cklee@phya.snu.ac.kr
[§]hsyang@phya.snu.ac.kr

Contents

1. Introduction

The Seiberg-Witten (SW) map, first formulated in [1], establishes an equivalence between noncommutative (NC) gauge theories and conventional gauge theories defined on ordinary (commutative) space. Consequently it becomes feasible to discuss several features of NC gauge theories in their commutative equivalents, thereby making the former more tractable. So far this analysis has been confined to source-free theories since the original map was given for the gauge potentials. In order to discuss NC gauge theories with sources it is therefore essential to have a corresponding map for the sources, which is otherwise lacking. One of the objectives of this paper is to provide such maps and also for the energy-momentum (EM) tensors.

A vexing issue is the apparent lack of agreement in the results obtained by first applying the map on the NC action with the source term added and then analyzing the equations of motion or, alternatively, by first obtaining the equations of motion in the NC version and then exploiting the map. These points were raised and discussed (for the source-free case) in [2, 3, 4] in various contexts. During the course of our analysis we show that, with proper interpretation, all disagreement or ambiguities are ironed out.

As stated earlier, we derive a map for the sources or the currents. This is a general result which can be expressed in a closed form. The map is explicitly worked out for the first nontrivial order in θ, which is the NC parameter. It is then used to relate the usual gauge invariant Adler-Bell-Jackiw (ABJ) anomaly [5] in the commutative case with the star gauge covariant anomaly in the NC theory. The interplay of anomaly with gauge invariance (or covariance) is also discussed. We then extend the analysis to provide a map for the energy-momentum tensors in the two descriptions, clarifying en route some subtleties in their definition. Along with the equations of motion in NC electrodynamics with sources, this yields the Lorentz force law.

It is our belief that the SW-type maps for the currents and EM tensors, aside from such familiar maps for the gauge and matter fields, deserve attention on its own right. Particularly, it allows one to discuss various physical aspects, irrespectively of either the detailed form or the SW maps for the matter fields. In this connection, we also discuss briefly their implications on an intriguing Sugawara-type formulation where the EM tensor is expressed sorely in terms of the currents. Compared to the dimension independent analysis in the rest of the paper, this part is confined to two dimensions.

In section 2, we derive the map for currents and anomalies. Section 3 contains the corresponding analysis for EM tensor and the derivation of the Lorentz force law in NC electrodynamics. Section 4 has, apart from the concluding remarks, a Sugawara-type construction in two dimensions which is also compatible with the results obtained in the previous sections.

2. Map for Currents and Anomalies

Here we derive a mapping of the currents in the NC and commutative descriptions. Also, this will be used to provide a map between the different anomalies. First, an algebraic approach is discussed where the results are given to the first order in the NC parameter, θ. This will be subsequently generalized, in a dynamical approach, to all orders in θ.

2.1 Algebraic approach

The original map [1, 6, 7] relating the gauge potentials and field tensors in NC $U(1)$ gauge theory [1],

$$\widehat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\alpha\beta} A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}) + \mathcal{O}(\theta^2), \tag{2.1}$$

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta}(F_{\mu\alpha}F_{\nu\beta} - A_\alpha \partial_\beta F_{\mu\nu}) + \mathcal{O}(\theta^2) \tag{2.2}$$

was obtained algebraically so that the stability of gauge transformations,

$$\widehat{\delta}_{\widehat{\lambda}}\widehat{A}_\mu \equiv \widehat{D}_\mu \star \widehat{\lambda} = \partial_\mu\widehat{\lambda} - i\widehat{A}_\mu \star \widehat{\lambda} + i\widehat{\lambda}\star\widehat{A}_\mu = \partial_\mu\widehat{\lambda} + \theta^{\alpha\beta}\partial_\alpha\widehat{A}_\mu\partial_\beta\widehat{\lambda} + \mathcal{O}(\theta^2), \tag{2.3}$$

$$\delta_\lambda A_\mu \equiv \partial_\mu\lambda, \tag{2.4}$$

may be insured by a further map among the gauge parameters

$$\widehat{\lambda} = \lambda + \frac{1}{2}\theta^{\alpha\beta}\partial_\alpha\lambda A_\beta + \mathcal{O}(\theta^2). \tag{2.5}$$

It may be noted that (2.2) is a consequence of (2.1), following from the basic definitions

$$\begin{aligned}\widehat{F}_{\mu\nu} &\equiv \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu - i\widehat{A}_\mu\star\widehat{A}_\nu + i\widehat{A}_\nu\star\widehat{A}_\mu \\ &= \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu + \theta^{\alpha\beta}\partial_\alpha\widehat{A}_\mu\partial_\beta\widehat{A}_\nu + \mathcal{O}(\theta^2)\end{aligned} \tag{2.6}$$

and

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.7}$$

[1] Variables in the NC space are distinguished from their conventional counterparts by a caret.

so that, whereas $F_{\mu\nu}$ is gauge invariant, $\widehat{F}_{\mu\nu}$ transforms covariantly under the star gauge transformation,

$$\widehat{\delta}_{\widehat{\lambda}}\widehat{F}_{\mu\nu} \equiv -i\widehat{F}_{\mu\nu}\star\widehat{\lambda} + i\widehat{\lambda}\star\widehat{F}_{\mu\nu} = \theta^{\alpha\beta}\partial_\alpha\widehat{F}_{\mu\nu}\partial_\beta\widehat{\lambda} + \mathcal{O}(\theta^2). \tag{2.8}$$

The proposed map among the currents $\widehat{J}^\mu$ and J^μ is now obtained under the following two conditions: the current J^μ is gauge invariant and satisfies the ordinary conservation law $\partial_\mu J^\mu = 0$, while the current $\widehat{J}^\mu$ transforms covariantly and satisfies the covariant conservation law $\widehat{D}_\mu\star\widehat{J}^\mu = 0$. Up to $\mathcal{O}(\theta)$, the stability under gauge transformations is easily attained by mimicking the map (2.2) among the field tensors,

$$\widehat{J}^\mu = J^\mu - \theta^{\alpha\beta}A_\alpha\partial_\beta J^\mu + \cdots, \tag{2.9}$$

where the ellipses indicate the freedom of adding more terms that are invariant under ordinary gauge transformations. It is clear that the most general structure is given by

$$\widehat{J}^\mu = J^\mu - \theta^{\alpha\beta}A_\alpha\partial_\beta J^\mu + c_1\theta^{\mu\alpha}F_{\alpha\beta}J^\beta + c_2\theta^{\alpha\beta}F_{\alpha\beta}J^\mu + c_3\theta^{\alpha\beta}F_\alpha{}^\mu J_\beta, \tag{2.10}$$

where c_1, c_2, and c_3 are undetermined coefficients. Next, demanding the simultaneous conservation $\widehat{D}_\mu\star\widehat{J}^\mu = \partial_\mu J^\mu = 0$ immediately fixes $c_1 = 2c_2 = 1$ and $c_3 = 0$ so that,

$$\begin{aligned}\widehat{J}^\mu &= J^\mu - \theta^{\alpha\beta}A_\alpha\partial_\beta J^\mu + \theta^{\mu\alpha}F_{\alpha\beta}J^\beta + \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta}J^\mu \\ &= J^\mu + (\theta FJ)^\mu + \partial_\alpha(\theta^{\alpha\beta}A_\beta J^\mu)\end{aligned} \tag{2.11}$$

where an obvious matrix notation has been introduced.

This yields the cherished map among the currents valid up to $\mathcal{O}(\theta)$. Observe that the derivation is based on general gauge transformation properties. The explicit structure of neither $\widehat{J}^\mu$ nor J^μ need to be specified. If any one of these is known, the other is determined through the map (2.11) or its inverse

$$J^\mu = \widehat{J}^\mu - (\theta\widehat{F}\widehat{J})^\mu - \partial_\alpha(\theta^{\alpha\beta}\widehat{A}_\beta\widehat{J}^\mu). \tag{2.12}$$

We now present a dynamical treatment which generalizes the above results, apart from precisely specifying the currents.

2.2 Dynamical approach

Let the noncommutative action be defined as

$$\begin{aligned}\widehat{S}(\widehat{A},\widehat{\psi}) &= -\frac{1}{4}\int d^4x\widehat{F}_{\mu\nu}\star\widehat{F}^{\mu\nu} + \widehat{S}_M(\widehat{\psi},\widehat{A}) \\ &\equiv \widehat{S}_{\rm ph}(\widehat{A}) + \widehat{S}_M(\widehat{\psi},\widehat{A})\end{aligned} \tag{2.13}$$

where the pure gauge term has been isolated in the "photonic" piece $\widehat{S}_{\rm ph}(\widehat{A})$. The charged matter fields are denoted by $\widehat{\psi}_\alpha$. Then the equation of motion for $\widehat{A}_\mu$ is

$$\frac{\delta\widehat{S}_{\rm ph}(\widehat{A})}{\delta\widehat{A}_\mu} = \widehat{D}_\nu\star\widehat{F}^{\nu\mu} = -\widehat{J}^\mu \tag{2.14}$$

where

$$\widehat{J}^\mu := \frac{\delta \widehat{S}_M(\widehat{\psi}, \widehat{A})}{\delta \widehat{A}_\mu}|_{\widehat{\psi}}. \tag{2.15}$$

Here, thanks to the equation of motion satisfied by $\widehat{\psi}_\alpha$, $\widehat{J}^\mu$ will satisfy the covariant conservation law

$$\widehat{D}_\mu \star \widehat{J}^\mu = 0. \tag{2.16}$$

This may also be seen by taking the covariant derivative on both sides of Eq. (2.14). The same equation also shows that $\widehat{J}^\mu$ transforms covariantly under the star gauge transformations. Clearly therefore, this $\widehat{J}^\mu$ is similar to the one considered in section 2.1.

Now we rewrite the action (2.13) using the SW map to obtain a $U(1)$ gauge invariant action defined on commutative space [8, 9] [2],

$$\widehat{S}(\widehat{A}, \widehat{\psi})|_{\text{SW map}} := S_{\text{ph}}(A) + S_M(\psi, A) \tag{2.17}$$

where $S_{\text{ph}}(A)$ contains all terms involving A^μ only and is given by

$$S_{\text{ph}}(A) = \int d^4x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}(F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu} - \frac{1}{4}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(\theta^2)\Big]. \tag{2.18}$$

In fact this action, modulo constant terms, is the expansion of the Born-Infeld action up to order $\mathcal{O}(F^3)$ (with $2\pi\alpha' = 1$) [1],

$$S_{BI} = \int d^4x \sqrt{-\det(\eta_{\mu\nu} - \theta_{\mu\nu} + F_{\mu\nu})}. \tag{2.19}$$

Now from Eq. (2.18) the gauge invariant equation of motion is obtained,

$$\frac{\delta S_{\text{ph}}(A)}{\delta A_\mu(x)} = -J^\mu(x) \tag{2.20}$$

where

$$J^\mu(x) := \frac{\delta S_M(\psi, A)}{\delta A_\mu(x)}|_\psi \tag{2.21}$$

which in general contains θ-dependent terms. Again, thanks to the equation of motion satisfied by ψ_α, J^μ now satisfies the ordinary conservation law,

$$\partial_\mu J^\mu = 0. \tag{2.22}$$

The same result is also inferred from the gauge invariance of Eqs. (2.18) and (2.20). This current, therefore, is similar to J^μ introduced in section 2.1.

It is now possible to obtain a relation between $\widehat{J}^\mu$ and J^μ by noticing that

$$\begin{aligned} \widehat{J}^\mu(x) &= \frac{\delta \widehat{S}_M(\widehat{\psi}, \widehat{A})}{\delta \widehat{A}_\mu}|_{\widehat{\psi}} \\ &\overset{\text{SW map}}{=} \int d^4y \Big[\frac{\delta S_M(A,\psi)}{\delta A_\nu(y)}|_\psi \frac{\delta A_\nu(y)}{\delta \widehat{A}_\mu(x)} + \frac{\delta S_M(A,\psi)}{\delta \psi_\alpha(y)}|_A \frac{\delta \psi_\alpha(y)}{\delta \widehat{A}_\mu(x)}\Big] \\ &= \int d^4y J^\nu(y) \frac{\delta A_\nu(y)}{\delta \widehat{A}_\mu(x)}, \end{aligned} \tag{2.23}$$

[2] A SW map for the matter sector is also necessary for this transition but its explicit structure is inconsequential for this analysis.

where the second term in the second line was dropped on using the equation of motion for ψ_α. Eq. (2.23) yields the general form of the map between the currents. Although it is displayed for four dimensions, the result is obviously valid for any dimensions.

As a simple yet nontrivial check, we now reproduce the $\mathcal{O}(\theta)$ result (2.11), starting from Eq. (2.23). From Eq. (2.1) it follows that

$$\frac{\delta}{\delta \widehat{A}_\mu(x)} = \frac{\delta}{\delta A_\mu(x)} + \int d^4y \theta^{\alpha\beta} \frac{\delta}{\delta A_\mu(x)} \Big(A_\alpha(y)\partial_\beta A_\lambda(y) - \frac{1}{2} A_\alpha(y) \partial_\lambda A_\beta(y) \Big) \frac{\delta}{\delta A_\lambda(y)} + \mathcal{O}(\theta^2). \tag{2.24}$$

Using this to evaluate the functional derivative in Eq. (2.23) immediately leads to Eq. (2.11) where, at an intermediate step, the current conservation (2.22) has been used.

The map (2.23) is also consistent with the observation,

$$\begin{aligned} -\widehat{J}^\mu(x) &= \widehat{D}_\nu \star \widehat{F}^{\nu\mu} = \frac{\delta \widehat{S}_{\rm ph}(\widehat{A})}{\delta \widehat{A}_\mu} \\ &\overset{\rm SW\ map}{=} \int d^4y \frac{\delta S_{\rm ph}(A)}{\delta A_\nu(y)} \frac{\delta A_\nu(y)}{\delta \widehat{A}_\mu(x)} = -\int d^4y J^\nu(y) \frac{\delta A_\nu(y)}{\delta \widehat{A}_\mu(x)}. \end{aligned} \tag{2.25}$$

It is also clear that the effective (non-linear) Maxwell equation with (gauge-invariant) source J^μ is naturally identified with the expression (2.20). Note, however, that this is in general different from the stationary condition obtained by applying the SW map to an action

$$\widehat{S}_J = \int d^4x \Big[-\frac{1}{4} \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu} + \widehat{A}_\mu \star \widehat{J}^\mu \Big], \tag{2.26}$$

although it also leads to the equation of motion (2.14). The discrepancy arises because the source term in Eq. (2.26) is not gauge invariant under NC $U(1)$ gauge transformations (with $\widehat{J}^\mu$ in the adjoint representation) so that the application of the SW map becomes meaningless. It is only after the inclusion of the full matter sector that gauge invariance is restored, leading to our original action (2.13).

As another consistency check on the construction (2.23) or (2.25), observe that the latter leads to an identity if everything is expressed in terms of the gauge potentials,

$$\widehat{D}_\nu \star \widehat{F}^{\nu\mu} \overset{\rm SW\ map}{=} \int d^4y \frac{\delta S_{\rm ph}(A)}{\delta A_\nu(y)} \frac{\delta A_\nu(y)}{\delta \widehat{A}_\mu(x)}, \tag{2.27}$$

where

$$\begin{aligned} \frac{\delta S_{\rm ph}(A)}{\delta A_\nu} &= \partial_\mu F^{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}\partial_\mu(F_{\alpha\beta}F^{\mu\nu}) - \frac{1}{4}\theta^{\mu\nu}\partial_\mu(F_{\alpha\beta}F^{\alpha\beta}) + \theta^{\nu\alpha}\partial_\mu(F_{\alpha\beta}F^{\beta\mu}) \\ &\quad -\theta^{\mu\alpha}\partial_\mu(F_{\alpha\beta}F^{\beta\nu}) + \theta_{\alpha\beta}\partial_\mu(F^{\alpha\mu}F^{\beta\nu}) + \mathcal{O}(\theta^2) \end{aligned} \tag{2.28}$$

is obtained from Eq. (2.18). Up to $\mathcal{O}(\theta)$ the left-hand side of Eq. (2.27) can be computed from a direct application of the SW map (2.1)-(2.2), leading to,

$$\widehat{D}_\mu \star \widehat{F}^{\mu\nu} \overset{\rm SW\ map}{=} \partial_\mu F^{\mu\nu} - \theta^{\alpha\beta} A_\alpha \partial_\beta \partial_\mu F^{\mu\nu} + \theta_{\alpha\beta}\partial_\mu(F^{\alpha\mu}F^{\beta\nu}) + \theta^{\alpha\mu}F_{\alpha\beta}\partial_\mu F^{\beta\nu}. \tag{2.29}$$

The right-hand side of Eq. (2.27) is next computed using Eqs. (2.28) and (2.24). After some algebra it reproduces Eq. (2.29) where the following identities were necessary

$$\frac{1}{2}\theta^{\alpha\beta}(\partial_\mu F_{\alpha\beta})F^{\mu\nu} + \theta^{\mu\alpha}(\partial_\mu F_{\alpha\beta})F^{\beta\nu} = 0, \tag{2.30}$$

$$-\frac{1}{4}\theta^{\mu\nu}\partial_\mu(F_{\alpha\beta}F^{\alpha\beta}) + \theta^{\nu\alpha}(\partial_\mu F_{\alpha\beta})F^{\beta\mu} = 0. \tag{2.31}$$

This proves the validity of the identity (2.27) at least to $\mathcal{O}(\theta)$.

The above analysis shows that consistent results are obtained irrespectively of whether the SW map is directly applied to the NC action or on the NC object obtained from the NC action. For the equation of motion, however, there is some subtlety which is next discussed.

An application of the SW map on the equation of motion (2.14) yields, on using Eqs. (2.29) and (2.11), the result

$$\begin{aligned}\widehat{W}^\nu \equiv\ & \partial_\mu\Big[(1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta})F^{\mu\nu} - (\theta F^2)^{\mu\nu} - (F\theta F)^{\mu\nu}\Big] + \theta^{\alpha\beta}\partial_\alpha\Big(A_\beta(\partial_\mu F^{\mu\nu} + J^\nu)\Big) \\ & + J^\nu + (\theta F J)^\nu = 0.\end{aligned} \tag{2.32}$$

On the other hand, the equation of motion (2.20) obtained after applying the map on the NC action (2.13) is given by

$$W^\nu \equiv \partial_\mu\Big[(1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta})F^{\mu\nu} - \frac{1}{4}\theta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - (\theta F^2)^{\mu\nu} - (F\theta F)^{\mu\nu} - (F^2\theta)^{\mu\nu}\Big] + J^\nu = 0. \tag{2.33}$$

The two equations (2.32) and (2.33) are not identical, leading to the suspicion that the implementation of the map is not a commutative operation [2, 3]. Why this difference occurs is not difficult to understand. The equation (2.32) was obtained from a gauge covariant equation of motion (2.14) while Eq. (2.33) was obtained from a gauge invariant one in Eq. (2.20). Nevertheless it is possible to establish a compatibility by calculating the difference

$$\widehat{W}^\nu - W^\nu = \theta^{\alpha\beta}\partial_\alpha\Big(A_\beta(\partial_\mu F^{\mu\nu} + J^\nu)\Big) + \theta^{\nu\alpha}F_{\alpha\beta}(\partial_\mu F^{\mu\beta} + J^\beta) \tag{2.34}$$

which follows easily on using the identity (2.31). Now it is seen from either Eq. (2.32) or Eq. (2.33) that the term in the parenthesis $(\partial_\mu F^{\mu\nu} + J^\nu)$ is at least of $\mathcal{O}(\theta)$. Hence $\widehat{W}^\nu = W^\nu$ up to the order we are dealing. This shows that the two equations of motion are compatible.

Finally we would like to mention that ambiguities [10, 11] in the basic SW map (2.1) do not affect the map (2.23) among the currents. Any two solutions may differ by a field dependent pure gauge $\partial_\mu\Lambda(A)$ which is also expected on general grounds since the SW transformation maps gauge equivalent classes. Under this difference we find from Eq. (2.23),

$$\Delta\widehat{J}^\mu(x) = \int d^4y J^\nu(y)\frac{\delta}{\delta\widehat{A}_\mu(x)}\Big(\partial_\nu\Lambda(A)\Big) = -\int d^4y\partial_\nu J^\nu(y)\frac{\delta\Lambda(A)}{\delta\widehat{A}_\mu(x)} = 0 \tag{2.35}$$

on using current conservation. Hence the map remains unchanged. This is similar to the map (2.2) which is also unaffected [11].

2.3 Anomalies and the map

The map for the currents found here also yields consistent results even if the current is anomalous - that is, its usefulness is not restricted to the strictly conserved or covariantly conserved currents. We show this for leading order in θ. First note that Eq. (2.11) can also be used to relate the axial currents $\widehat{J}_5^\mu$ and J_5^μ at the classical (tree) level. This is because, in that case, these currents satisfy the same gauge transformation properties and conservation laws as for the corresponding vector currents. The issue is more subtle at the quantum level where, due to the one loop effects, simultaneous conservation of J^μ and J_5^μ is not possible [5]. To fix our notions we take J_5^μ to be anomalous. Since $\partial_\mu J_5^\mu$ no longer vanishes, it is natural to think that Eq. (2.11) may be modified such that it contains an extra $\mathcal{O}(\theta)$-term, proportional to $\partial_\mu J_5^\mu$, in its right hand side. But, as long as we insist that $\widehat{J}_5^\mu$ be $\star$-gauge covariant and J_5^μ be gauge-invariant, the extra term should be gauge invariant by itself. (In this regard, see Eq. (2.10)). However, using $\theta^{\alpha\beta}$, $F^{\mu\nu}$, and $\partial_\nu J_5^\nu$, no such gauge invariant term (with correct dimension and appropriate tensor structure) can be found. Hence we expect our formula (2.11) to apply even for this anomalous case.

Given the relation (2.11), taking its covariant divergence yields

$$\widehat{D}_\mu \star \widehat{J}^\mu = \partial_\mu J^\mu + \theta^{\alpha\beta}\partial_\alpha(A_\beta \partial_\mu J^\mu). \tag{2.36}$$

What was discussed till now ($\partial_\mu J^\mu = \widehat{D}_\mu \star \widehat{J}^\mu = 0$) is obviously compatible with the above relation. Let us now consider the anomalous case (where, for notational simplicity, $\widehat{J}^\mu$ stands for axial current) for which we have [12]

$$\widehat{D}_\mu \star \widehat{J}^\mu = N \star (\widehat{F} \wedge \widehat{F} \wedge \cdots \wedge \widehat{F})_{\text{n-fold}}. \tag{2.37}$$

The right hand side here is the (star) gauge covariant anomaly in $d = 2n$ dimensions, with N being the normalization and using the (star) wedge notation

$$\star(\widehat{F} \wedge \cdots \wedge \widehat{F}) = \varepsilon_{\mu\nu\cdots\lambda\rho}\widehat{F}^{\mu\nu} \star \cdots \star \widehat{F}^{\lambda\rho}. \tag{2.38}$$

Up to $\mathcal{O}(\theta)$ the star products involving $\widehat{F}$ can be replaced by ordinary products so that, after applying the SW map (2.2), the anomaly (2.37) reduces to,

$$\widehat{D}_\mu \star \widehat{J}^\mu \overset{\text{SW\_map}}{=} N\Big[(F \wedge F \wedge \cdots \wedge F) - n(F\theta F) \wedge (F \wedge \cdots \wedge F) - \theta^{\alpha\beta}A_\alpha \partial_\beta (F \wedge \cdots \wedge F)\Big]. \tag{2.39}$$

Using the identity [13],

$$\theta^{\alpha\beta}F_{\alpha\beta}(F \wedge \cdots \wedge F) = -2n(F\theta F) \wedge (F \wedge \cdots \wedge F), \tag{2.40}$$

we then get

$$\widehat{D}_\mu \star \widehat{J}^\mu \overset{\text{SW\_map}}{=} N\Big[F \wedge F \wedge \cdots \wedge F + \theta^{\alpha\beta}\partial_\alpha(A_\beta F \wedge \cdots \wedge F)\Big]. \tag{2.41}$$

Comparing this with Eq. (2.36) the usual gauge invariant anomaly in the SW deformed theory is deduced, i.e.,

$$\partial_\mu J^\mu = N(F \wedge F \wedge \cdots \wedge F) \tag{2.42}$$

which is the expected result. Indeed the fact that the standard ABJ-anomaly is not modified in θ-expanded gauge theory was earlier shown in [14]. (For a mapping of the gauge invariant anomaly in either description, see [15, 13].) It appears, therefore, that our map (2.11) correctly incorporates quantum effects.

As another application, it is possible to discuss the shift in the gauge invariance, exactly as happens in the commutative case. Although it is possible, as before, to analyze in arbitrary dimensions, we confine to $d = 4$ where the usual ABJ-anomaly is

$$\partial_\mu J^\mu = \frac{1}{16\pi^2}\varepsilon_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}. \tag{2.43}$$

Defining a modified current as

$$\widetilde{J}^\mu = J^\mu - \frac{1}{8\pi^2}\varepsilon^{\mu\nu\lambda\rho}A_\nu F_{\lambda\rho} \tag{2.44}$$

leads to an anomaly free ($\partial_\mu \widetilde{J}^\mu = 0$) but gauge noninvariant current [16]. To do a similar thing for the NC case, rewrite the map (2.11) by replacing J^μ in favor of $\widetilde{J}^\mu$. The $\widetilde{J}^\mu$ independent terms are then moved to the other side and a new $\widehat{\widetilde{J}}^\mu$ is defined as

$$\widehat{\widetilde{J}}^\mu = \widehat{J}^\mu + \widehat{X}^\mu(\widehat{A}), \tag{2.45}$$

so that

$$\widehat{\widetilde{J}}^\mu = \widetilde{J}^\mu + (\theta F\widetilde{J})^\mu + \partial_\alpha(\theta^{\alpha\beta}A_\beta\widetilde{J}^\mu). \tag{2.46}$$

Note that all A_μ-dependent terms lumped in $\widehat{X}^\mu$ can be recast in terms of $\widehat{A}_\mu$ using the SW map. Since Eq. (2.46) is structurally identical to Eq. (2.11), a relation akin to Eq. (2.36) follows,

$$\widehat{D}_\mu \star \widehat{\widetilde{J}}^\mu = \partial_\mu\widetilde{J}^\mu + \theta^{\alpha\beta}\partial_\alpha(A_\beta\partial_\mu\widetilde{J}^\mu). \tag{2.47}$$

Since $\partial_\mu\widetilde{J}^\mu = 0$ it follows that $\widehat{D}_\mu \star \widehat{\widetilde{J}}^\mu = 0$. We are thereby successful in constructing an anomaly free current which however does not transform covariantly. Its lack of covariance is caused by the $\widehat{X}^\mu$ term in Eq. (2.45), which plays a role analogous to the Chern-Simons three form in the usual commutative description.

3. Energy-Momentum Tensors and Lorentz Force Law

The problems of defining EM tensors in NC gauge theories have been studied by various authors [17, 2, 3, 4, 18] but the results have not always agreed. In this section a systematic presentation is done which naturally leads to a map among these tensors in the different (NC and commutative) descriptions. A fall out of the analysis is the Lorentz force law in NC space. As usual, the Lorentz force is identified through considering the 4-divergence of electromagnetic EM tensor.

To define a manifestly symmetric electromagnetic EM tensor on NC space, the NC gauge fields are formally coupled to a weak external gravitational field

$$\widehat{S}_{\widehat{g}} = -\frac{1}{4}\int d^4x\sqrt{-\widehat{g}} \star \widehat{g}^{\mu\lambda} \star \widehat{g}^{\nu\rho} \star \widehat{F}_{\mu\nu} \star \widehat{F}_{\lambda\rho}. \tag{3.1}$$

The EM tensor is defined as

$$\widehat{T}_{\mu\nu} = \frac{2}{\sqrt{-\widehat{g}}}\frac{\delta \widehat{S}_{\widehat{g}}}{\delta \widehat{g}^{\mu\nu}}|_{\widehat{g}^{\mu\nu}=\eta^{\mu\nu}}. \tag{3.2}$$

There may be an ordering ambiguity in the above manipulation, but that is inconsequential since eventually the metric is set flat. We find

$$\widehat{T}_{\mu\nu} = \frac{1}{2}(\widehat{F}_{\mu\lambda}\star\widehat{F}^{\lambda}{}_{\nu} + \widehat{F}_{\nu\lambda}\star\widehat{F}^{\lambda}{}_{\mu}) + \frac{1}{4}\eta_{\mu\nu}\widehat{F}_{\lambda\rho}\star\widehat{F}^{\lambda\rho}. \tag{3.3}$$

This tensor is both symmetric and traceless. However it is not star gauge invariant. Rather, it is star gauge covariant. Expectedly, a covariant conservation law holds,

$$\widehat{D}_{\mu}\star\widehat{T}^{\mu\nu} = 0 \tag{3.4}$$

which follows on using the source free equation of motion (see Eq. (2.14)) and the (NC) Bianchi identity

$$\widehat{D}_{\mu}\star\widehat{F}_{\nu\lambda} + \widehat{D}_{\nu}\star\widehat{F}_{\lambda\mu} + \widehat{D}_{\lambda}\star\widehat{F}_{\mu\nu} = 0. \tag{3.5}$$

Now the EM tensor $T_{\mu\nu}$ in commutative space is gauge invariant and satisfies the ordinary conservation law. From an algebraic point of view, therefore, $\widehat{T}_{\mu\nu}$ and $T_{\mu\nu}$ (for each given ν) simulate exactly the roles of the sources $\widehat{J}^{\mu}$ and J^{μ}. It is not unreasonable to expect that the EM tensors also satisfy a map analogous to Eq. (2.11), i.e., up to $\mathcal{O}(\theta)$,

$$\widehat{T}^{\mu\nu} = T^{\mu\nu} + (\theta F T)^{\mu\nu} + \partial_{\alpha}(\theta^{\alpha\beta}A_{\beta}T^{\mu\nu}). \tag{3.6}$$

We now prove that this is indeed so, simultaneously fixing the structure of $T^{\mu\nu}$.

Before proceeding further it may be pointed out that Eq. (3.3) also follows from a Noether procedure involving the combination of translations with field dependent gauge transformations [19]. Explicitly, acting the generator

$$\widehat{W}^{T}_{\mu} = \frac{1}{2}\int d^4x(\widehat{F}_{\mu\nu}\star\frac{\delta}{\delta\widehat{A}_{\nu}} + \frac{\delta}{\delta\widehat{A}_{\nu}}\star\widehat{F}_{\mu\nu}) \tag{3.7}$$

on the flat NC action $\widehat{S}_{\rm flat}$ gives rise to

$$\begin{aligned}\widehat{W}^{T}_{\mu}\widehat{S}_{\rm flat} &= -\int d^4x\widehat{D}^{\nu}\star\widehat{T}_{\mu\nu}\\ &= -\int d^4x\widehat{D}^{\nu}\star\Big(\frac{1}{2}(\widehat{F}_{\mu\lambda}\star\widehat{F}^{\lambda}{}_{\nu} + \widehat{F}_{\nu\lambda}\star\widehat{F}^{\lambda}{}_{\mu}) + \frac{1}{4}\eta_{\mu\nu}\widehat{F}_{\lambda\rho}\star\widehat{F}^{\lambda\rho}\Big),\end{aligned} \tag{3.8}$$

where we have used the identity (3.5).

Now expanding the EM tensor in Eq. (3.3) up to the leading order in θ, by using Eq. (2.2), yields

$$\begin{aligned}\widehat{T}_{\mu\nu}|_{\rm SW\ map} &= (1-\frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta})\Big((F^2)_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}F^2\Big) - (F^2\theta F + F\theta F^2)_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}{\rm Tr}(F\theta F^2)\\ &\quad +\partial_{\alpha}\Big[\theta^{\alpha\beta}A_{\beta}\Big((F^2)_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}F^2\Big)\Big]\\ &= \Big[(1-\frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta})F_{\mu\lambda} - \frac{1}{4}\theta_{\mu\lambda}F^2 - (F^2\theta + F\theta F + \theta F^2)_{\mu\lambda}\Big]F^{\lambda}{}_{\nu} - \eta_{\mu\nu}\mathcal{L}_{\rm ph}\\ &\quad +(\theta F^3)_{\mu\nu} + \frac{1}{4}(\theta F)_{\mu\nu}F^2 + \partial_{\alpha}(\theta^{\alpha\beta}A_{\beta}T^0_{\mu\nu})\\ &= \Pi_{\mu\lambda}F^{\lambda}{}_{\nu} - \eta_{\mu\nu}\mathcal{L}_{\rm ph} + (\theta F T^0)_{\mu\nu} + \partial_{\alpha}(\theta^{\alpha\beta}A_{\beta}T^0_{\mu\nu})\end{aligned} \tag{3.9}$$

where $\mathcal{L}_{\rm ph}$ is the Lagrangian density for nonlinear photons read off from Eq. (2.18) and $T^0_{\mu\nu}$ is the EM tensor for $\theta = 0$,

$$T^0_{\mu\nu} = (F^2)_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}F^2 \tag{3.10}$$

while $\Pi_{\mu\nu}$ is the generalized canonical momenta as defined by

$$\Pi_{\mu\nu} = -\frac{\partial \mathcal{L}_{\rm ph}}{\partial(\partial^\mu A^\nu)} = (1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta})F_{\mu\nu} - \frac{1}{4}\theta_{\mu\nu}F^2 - (F^2\theta + F\theta F + \theta F^2)_{\mu\nu}. \tag{3.11}$$

The EM tensor in the commutative picture is likewise obtained from the operator analogous to that in Eq. (3.7), i.e., using the generator [19]

$$W^T_\mu = \int d^4x F_{\mu\nu}\frac{\delta}{\delta A_\nu}, \tag{3.12}$$

and the relation

$$W^T_\mu S_{\rm ph} = -\int d^4x \partial^\nu T_{\mu\nu} \tag{3.13}$$

where $S_{\rm ph}$ is defined in Eq. (2.18), so that we find

$$T_{\mu\nu} = \Pi_{\mu\lambda}F^\lambda{}_\nu - \eta_{\mu\nu}\mathcal{L}_{\rm ph}. \tag{3.14}$$

Using the free field equation of motion $\partial^\mu\Pi_{\mu\nu} = 0$ (which follows from Eq. (2.33) by setting $J_\nu = 0$) and

$$\partial^\nu\mathcal{L}_{\rm ph} - \frac{\partial\mathcal{L}_{\rm ph}}{\partial(\partial_\mu A_\lambda)}\partial^\nu(\partial_\mu A_\lambda) = 0, \tag{3.15}$$

it is easy to see that

$$\partial^\mu T_{\mu\nu} = 0. \tag{3.16}$$

Since this EM tensor was obtained from the commutative equivalent of the NC theory, it is the one that should be used in the map. Furthermore it is reassuring to note that $T_{\mu\nu}$ is both gauge invariant and conserved, exactly as desired. Now $T^0_{\mu\nu}$ in Eq. (3.10) and $T_{\mu\nu}$ in Eq. (3.14) differ by terms of $\mathcal{O}(\theta)$, so that Eq. (3.9) may be cast precisely in the form (3.6). This completes the derivation, up to $\mathcal{O}(\theta)$, of the map between the EM tensors. Note that $T^{\mu\nu}$ appearing in the map is neither symmetric nor traceless. This is due to the fact that Lorentz and classical conformal invariance are broken in NC electrodynamics [9].

Inclusion of sources does not pose any problem. The structures of the relevant electromagnetic EM tensors remain the same, but the conservation laws in Eqs. (3.4) and (3.16) are modified leading to the respective Lorentz force laws.

Starting from Eq. (3.3) and using the equation of motion (2.14) together with the Bianchi identity (3.5) immediately yields the NC generalization of the Lorentz force law

$$\widehat{D}_\mu \star \widehat{T}^{\mu\nu} = -\frac{1}{2}(\widehat{J}_\mu \star \widehat{F}^{\mu\nu} + \widehat{F}^{\mu\nu} \star \widehat{J}_\mu). \tag{3.17}$$

Similarly, the corresponding law in the commutative picture emerges by considering the equation of motion (2.20) and takes the form

$$\partial_\mu T^{\mu\nu} = -J_\mu F^{\mu\nu}. \tag{3.18}$$

As a consistency of the whole program, we show that the deformation in the Lorentz force law as given by Eq. (3.17) is compatible with Eq. (3.18). Using the expressions for the various maps, it turns out that, up to $\mathcal{O}(\theta)$,

$$\widehat{D}_\mu \star \widehat{T}^{\mu\nu}|_{\text{SW map}} = \partial_\mu T^{\mu\nu} + \partial_\alpha(\theta^{\alpha\beta} A_\beta \partial_\mu T^{\mu\nu}) \tag{3.19}$$

and,

$$\frac{1}{2}(\widehat{J}_\mu \star \widehat{F}^{\mu\nu} + \widehat{F}^{\mu\nu} \star \widehat{J}_\mu)|_{\text{SW map}} = J_\mu F^{\mu\nu} + \partial_\alpha(\theta^{\alpha\beta} A_\beta J_\mu F^{\mu\nu}). \tag{3.20}$$

Adding them together yields,

$$\begin{aligned}\Big[\widehat{D}_\mu \star \widehat{T}^{\mu\nu} + \frac{1}{2}(\widehat{J}_\mu \star \widehat{F}^{\mu\nu} + \widehat{F}^{\mu\nu} \star \widehat{J}_\mu)\Big]_{\text{SW map}} &= \\ \partial_\mu T^{\mu\nu} + J_\mu F^{\mu\nu} + \partial_\alpha\Big(\theta^{\alpha\beta} A_\beta(\partial_\mu T^{\mu\nu} + J_\mu F^{\mu\nu})\Big)&. \end{aligned} \tag{3.21}$$

It is now clear that Eq. (3.18) implies Eq. (3.17). Incidentally Eq. (3.19) is the exact analogue of Eq. (2.36) that maps the source divergence.

4. Discussion

We have provided a Seiberg-Witten like map relating the sources in the noncommutative (NC) and commutative descriptions. With its help, a commutative equivalent of NC electrodynamics with sources was formulated. Consistent results were obtained by applying the map either on the action or on the equations of motion. Although the map could, in principle, be worked to higher orders in θ (the NC parameter), for reasons of compactness $\mathcal{O}(\theta)$ results were explicitly analyzed. In this regime the map was also used to relate the star gauge covariant anomaly in the NC theory with the gauge invariant ABJ-anomaly in the θ-deformed theory.

Our methods were then extended to reveal a mapping among the energy-momentum (EM) tensors in the two descriptions. In the presence of sources, the NC generalization of the Lorentz force law was derived. The various maps were used to show that the deformation of Lorentz force law was consistent in the sense that enforcing this law in the commutative picture automatically enforced it in the NC picture.

Despite the different methods and different variables (e.g. currents, EM tensors, etc) used, a universal structure seemed to emerge in the various maps, at least to $\mathcal{O}(\theta)$. This reinforces the role of gauge transformations in mapping variables in NC gauge theories with their commutative equivalents.

As yet another manifestation of this universality, we discuss, for the special case of two dimensions, a Sugawara-type construction where EM tensors are expressed in terms of currents. In two dimensions the NC parameter $\theta^{\mu\nu} = \theta\varepsilon^{\mu\nu}$ really transforms as a Lorentz tensor so that invariances or symmetries not valid in higher dimensions may be restored in this case. This leads to a viability of alternative formulations where the EM tensor is symmetric. It may be recalled that even in commutative field theory, two dimensions play a special role with properties like exact solvability, bosonization, etc.

We begin with the commutative theory. Here it is known [20] that the EM tensor of a conformally invariant theory is expressed solely in terms of the currents,

$$T_{\mu\nu} = \frac{\pi}{2}(J_\mu J_\nu + J_\nu J_\mu - \eta_{\mu\nu} J_\lambda J^\lambda) \tag{4.1}$$

which is referred as the Sugawara form. Then, in the NC theory context, we may consider a natural noncommuatative generalization of this form, i.e.,

$$\widehat{T}_{\mu\nu} = \frac{\pi}{2}(\widehat{J}_\mu \star \widehat{J}_\nu + \widehat{J}_\nu \star \widehat{J}_\mu - \eta_{\mu\nu} \widehat{J}_\lambda \star \widehat{J}^\lambda). \tag{4.2}$$

Now the EM tensor of the commutative equivalent of this NC theory can be obtained using our map (3.6), together with the current map (2.11). A surprise is that, for this EM tensor, we find back the form (4.1); but, of course, J^μ can contain θ-dependent corrections here. This is demonstrated below.

Expanding the star product in Eq. (4.2) yields,

$$\widehat{T}_{\mu\nu} = \frac{\pi}{2}(\widehat{J}_\mu \widehat{J}_\nu + \widehat{J}_\nu \widehat{J}_\mu - \eta_{\mu\nu} \widehat{J}_\lambda \widehat{J}^\lambda) + \frac{i\pi}{4}\theta^{\alpha\beta}(\partial_\alpha \widehat{J}_\mu \partial_\beta \widehat{J}_\nu + \partial_\alpha \widehat{J}_\nu \partial_\beta \widehat{J}_\mu). \tag{4.3}$$

In the second parentheses, the NC variable can be replaced by the commutative one, since the analysis is done up to $\mathcal{O}(\theta)$. Then it can be expressed as a commutator $\theta^{\alpha\beta}\partial_\alpha\partial_\beta[J_\mu, J_\nu]$[3] which vanishes from symmetry arguments. Now inserting the map (2.11) in Eq. (4.3) leads to,

$$\widehat{T}_{\mu\nu} = T_{\mu\nu} + \text{order } \theta \text{ terms} \tag{4.4}$$

where $T_{\mu\nu}$ is defined in Eq. (4.1). After a slightly lengthy algebra, we get

$$\widehat{T}_{\mu\nu} = T_{\mu\nu} + 2(\theta F T)_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\beta} T_{\mu\nu} + \theta^{\alpha\beta} A_\beta \partial_\alpha T_{\mu\nu}. \tag{4.5}$$

Using the identity,

$$(\theta F T)_{\mu\nu} = \theta_{\mu\alpha} F^{\alpha\beta} T_{\beta\nu} = -\frac{1}{2}\theta^{\alpha\beta} F_{\alpha\beta} T_{\mu\nu}, \tag{4.6}$$

the equation (4.5) then reduces to

$$\widehat{T}_{\mu\nu} = T_{\mu\nu} + (\theta F T)_{\mu\nu} + \partial_\alpha(\theta^{\alpha\beta} A_\beta T_{\mu\nu}). \tag{4.7}$$

Since this has an identical structure as Eq. (3.6), we now conclude that $T^{\mu\nu}$ as given by Eq. (4.1) is the full expression to $\mathcal{O}(\theta)$. Incidentally, contrary to the earlier case, here both $\widehat{T}_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric because $\theta^{\mu\nu}$ in two dimensions is invariant under Lorentz transformations. Also it appears that, at least to order θ, the scale invariance is preserved.

Finally, from a general point of view, we end with the following remarks: the fact that anomalies could be related (Section 2.3), strongly suggests the feasibility of obtaining SW-type maps for effective actions. These would find an obvious application of connecting consistent as well as covariant anomalies for $U(N)$ gauge theories in the two descriptions. Presumably trace anomalies related to the EM tensors could also be discussed within this formulation. These topics are left for the future.

[3] Actually all products of currents have to be properly interpreted by a point-splitting regularization [20] in which case $[J_\mu(x), J_\nu(y)]$ is just a function of $(x - y)$. Indeed, to give a definite meaning to the Sugawara construction, such a prescription is implicitly assumed.

Acknowledgments

The work of RB was supported by funds from Sungkyunkwan University and Seoul National University. He is also grateful to the members of the respective physics departments for their gracious hospitality. The work of CL was supported by Korea Science Foundation ABRL program (R14-2003-012-01002-0) and by a 2003 Interdisciplinary Research Grant of Seoul National University. HSY was supported by the Brain Korea 21 Project in 2003.

References

[1] N. Seiberg and E. Witten, *J. High Energy Phys.* **09** (1999) 032, `hep-th/9908142`.

[2] S. I. Kruglov, *Ann. Fond. Louis Broglie* **27** (2002) 343, `hep-th/0110059`.

[3] J. M. Grimstrup, B. Kloibock, L. Popp, V. Putz, M. Schweda, and M. Wickenhauser, The Energy-Momentum Tensor in Noncommutative Gauge Field Models, `hep-th/0210288`.

[4] A. Das and J. Frenkel, *Phys. Rev.* **D 67** (2003) 067701, `hep-th/0212122`.

[5] S. L. Adler, *Phys. Rev.* **177** (1996) 2426; J. S. Bell and R. Jackiw, *Nuovo Cim.* **60A** (1969) 47.

[6] J. Madore, S. Schraml, P. Schupp, and J. Wess, *Eur. Phys. J.* **C 16** (2000) 161, `hep-th/0001203`.

[7] B. Jurčo, L. Möller, S. Schraml, P. Schupp, and J. Wess, *Eur. Phys. J.* **C 21** (2001) 383, `hep-th/0104153`.

[8] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda, and R. Wulkenhaar, Deformed QED via Seiberg-Witten map, `hep-th/0102103`.

[9] S. M. Carroll, J. A. Harvey, V. A. Kostelecký, C. D. Lane, and T. Okamoto, *Phys. Rev. Lett.* **87** (2001) 141601, `hep-th/0105082`.

[10] T. Asakawa and I. Kishimoto, *J. High Energy Phys.* **11** (1999) 024, `hep-th/9909139`.

[11] S. Fidanza, *J. High Energy Phys.* **06** (2002) 016, `hep-th/0112027`.

[12] J. M. Gracia-Bondia and C. P. Martin, *Phys. Lett.* **B 479** (2000) 321, `hep-th/0002171`; F. Ardalan and N. Sadooghi, *Int. J. Mod. Phys.* **A 16** (2001) 3151, `hep-th/0002143`.

[13] R. Banerjee, Anomalies in noncommutative gauge theories, Seiberg-Witten transformation and Ramond-Ramond couplings, to appear in *Int. J. Mod. Phys.* **A**, `hep-th/0301174`.

[14] F. Brandt, C. P. Martin, and F. Ruiz Ruiz, *J. High Energy Phys.* **07** (2003) 068, `hep-th/0307292`.

[15] R. Banerjee and S. Ghosh, *Phys. Lett.* **B 533** (2002) 162, `hep-th/0110177`.

[16] W. A. Bardeen, *Phys. Rev.* **184** (1969) 1848.

[17] M. Abou-Zeid and H. Dorn, *Phys. Lett.* **B 514** (2001) 183, `hep-th/0104244`.

[18] S. Ghosh, Space-time Symmetries in Noncommutative Gauge Theory: A Hamiltonian Analysis, `hep-th/0310155`.

[19] R. Jackiw, *Phys. Rev. Lett.* **41** (1978) 1635; R. Jackiw and S.-Y. Pi, *Phys. Rev. Lett.* **88** (2002) 111603, `hep-th/0111122`.

[20] E. Abdalla, M. C. B. Abdalla, and K. D. Rothe, *Non-pertubative Methods in 2 Dimensional Quantum Field Theory*, 2nd ed. (World Scientific, Singapore, 2001).